

Design of two-dimensional spatially coupled LDPC codes for combating burst erasures^①

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Abstract

Spatially-coupled low-density parity-check (SC-LDPC) codes are prominent candidates for future communication standards due to their 'threshold saturation' properties. However, when facing burst erasures, the decoding process will stop and the decoding performances will dramatically degrade. To improve the ability of burst erasure corrections, this paper proposes a two-dimensional SC-LDPC (2D-SC-LDPC) codes constructed by parallelly connecting two asymmetric SC-LDPC coupled chains for resistance to burst erasures. Density evolution algorithm is presented to evaluate the asymptotic performances against burst erasures, by which the maximum correctable burst erasure length can be computed. The analysis results show that the maximum correctable burst erasure lengths of the proposed 2D-SC-LDPC codes are much larger than the SC-LDPC codes and the asymmetric SC-LDPC codes. Finite-length performance simulation results of the 2D-SC-LDPC codes over the burst erasure channel confirm the excellent asymptotic performances.

Key words: burst erasure channel, SC-LDPC code, density evolution, decoding performance

0 Introduction

Burst erasures occur in many common practical communication scenarios, such as deep fading in wireless communications, intense transient noises in magnetic data storage systems, packet losses in Internet transmission systems, collisions in coded slotted ALOHA and so on, which will dramatically affect the communication reliability. Designing a good error-correcting code to mitigate the adverse effects of burst erasures is still a challenging problem.

Recently, spatially-coupled low-density parity-check (SC-LDPC) codes have attracted much attention since their belief propagation (BP) thresholds can achieve the maximum a posteriori (MAP) thresholds of the underlying LDPC block codes. Refs [1, 2] named this phenomenon 'threshold saturation' and proved it rigorously for the binary erasure channel (BEC)^[1] and the binary memoryless symmetric (BMS) channel^[2]. Afterwards, many new SC-LDPC code structures have emerged to further improve the thresholds^[3-4]. However, when facing burst erasures, the decoding chain of

SC-LDPC codes will break up and the decoding error probability will remain strictly positive from the position where the burst erasures occur. In other words, the excellent decoding performances of SC-LDPC codes will degrade seriously when transmitted over the burst erasure channel. More detailed analysis on the burst erasure correction capabilities of SC-LDPC codes can be found in Refs [5, 6] and finite-length analysis over the burst erasure channels was done in Ref. [7].

To combat burst erasures, multi-dimensional SC-LDPC codes constructed by expanding the one-dimensional edge-spreading to multiple dimensions were proposed in Ref. [8] and the analysis results showed that multi-dimensional SC-LDPC codes are more robust against the burst erasures compared with the conventional SC-LDPC codes. In Ref. [9], a band-splitting permutation for SC-LDPC codes was proposed to realize the asymptotic optimality in terms of burst erasure correction. Asymmetric SC-LDPC (ASC-LDPC) codes against multiple-burst erasures was proposed in Ref. [10], which proved that the multiple-burst erasure correcting performances of SC-LDPC codes can be greatly improved by introducing the asymmetric struc-

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The basis matrix \mathbf{B}_{2D} of a (J_1, K_1, J_2, K_2, L) 2D-SC-LDPC coupled chain is

$$\mathbf{B}_{2D} = \begin{bmatrix} \mathbf{B}'_{ASC, 1} & \mathbf{L}_2 \\ \mathbf{L}_1 & \mathbf{B}'_{ASC, 2} \end{bmatrix} \quad (6)$$

where $\mathbf{B}'_{ASC, k}$ denotes the remaining matrix after subscribing the exchanging edges from the base matrix $\mathbf{B}_{ASC, k}$, and \mathbf{L}_k denotes the interconnections between these two coupled chains, $k = 1, 2$.

The design rate of a $(J_1, K_1, J_2, K_2, L, M)$ 2D-SC-LDPC code is given as follows.

$$R_{2D} = 1 - \frac{(2L + 2\omega - 1)J'M}{2LK'M} \quad (7)$$

2 Density evolution analysis

This section analyzes the thresholds of the proposed 2D-SC-LDPC codes transmitted over the burst erasure channel under BP decoding algorithm. Consider that the single and consecutive burst erasure model and define the threshold as the maximum correctable burst erasure length. Here, a modified density evolution algorithm is proposed to compute the threshold and denote the threshold is denoted as e_{BP} .

When a 2D-SC-LDPC code is transmitted over the burst erasure channel under the single and consecutive burst erasure model, it means that the messages within the range of the burst erasures will be erased with probability 1 and the remained messages will be received correctly. For a clear description, the following symbols are defined.

$p_{i, k}^{(\ell)}$ denotes the erasure probability of the message from the variable node at position i of the k th chain to the check node at the l th iteration, where $k = 1, 2$ and $i \in \{1, 2, \dots, L\}$.

$q_{i, k}^{(\ell)}$ denotes the erasure probability of the message from the check node at position i of the k th chain to the variable node at the l th iteration.

Next, the updated equations of the density evolution for the l th iteration are derived.

Firstly, the initial error probability is given in Eq. (8), where e_{start} and e_{end} denote the start position and the end position of the burst erasures. Denote the burst erasure length as e_{len} and $e_{len} = e_{end} - e_{start}$.

$$p_{i, k}^{(0)} = \begin{cases} 1 & e_{start} \leq i \leq e_{end} \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Secondly, update for the check nodes. The computation of $q_{i, k}^{(\ell)}$ consists of two parts: one is for the selected check nodes with the fraction n/J'_k and the other is for the non-selected check nodes with the fraction $1 - n/J'_k$. Therefore, the calculation of $q_{i, k}^{(\ell)}$ is given in Eq. (9).

$$q_{i, k}^{(\ell)} = \left(1 - \frac{n}{J'_k}\right) \times \left(1 - \left(1 - \frac{1}{\omega} \left(\frac{1}{2} \sum_{b=0}^{\omega-1} p_{i-2b, k}^{(\ell-1)} + \frac{1}{2} \sum_{b=0}^{\omega-1} p_{i-b, k}^{(\ell-1)}\right)\right)^{K_k-1}\right) + \frac{n}{J'_k} \times \left(1 - \left(1 - \frac{1}{\omega} \left(\frac{1}{2} \left(\sum_{\omega-1} p_{i-2b, k}^{(\ell-1)} + p_{i-2, z}^{(\ell-1)}\right) + \frac{1}{2} \left(\sum_{\omega-1} p_{i-b, k}^{(\ell-1)} + p_{i-1, z}^{(\ell-1)}\right)\right)\right)^{K_k-1}\right) \quad (9)$$

where $z = (k \bmod 2) + 1$.

Finally, update for the variable nodes. Similarly, the computation of $p_{i, k}^{(\ell)}$ also consists of two parts: one is for the selected variable nodes with the fraction m/K'_k and the other is the non-selected variable nodes with the fraction $1 - m/K'_k$. The calculation of $p_{i, k}^{(\ell)}$ is given by

$$p_{i, k}^{(\ell)} = \left(1 - \frac{m}{K'_k}\right) \times p_{i, k}^{(0)} \left(\frac{1}{\omega} \left(\frac{1}{2} \sum_{a=0}^{\omega-1} q_{i+2a, k}^{(\ell)} + \frac{1}{2} \sum_{a=0}^{\omega-1} q_{i+a, k}^{(\ell)}\right)\right)^{J_k-1} + \frac{m}{K'_k} \times p_{i, k}^{(0)} \left(\frac{1}{\omega} \left(\frac{1}{2} \left(\sum_{\substack{a \neq 0 \\ a=1}}^{\omega-1} q_{i+2a, k}^{(\ell)} + q_{i+2, z}^{(\ell)}\right) + \frac{1}{2} \left(\sum_{\substack{a \neq 1 \\ a=1}}^{\omega-1} q_{i+a, k}^{(\ell)} + q_{i+1, z}^{(\ell)}\right)\right)\right)^{J_k-1} \quad (10)$$

The threshold e_{BP} can be defined as the maximum correctable burst erasure length to ensure that $p_{i, k}^{(\ell)}$ converges to zero for all variable nodes when l tends to infinity.

3 Numerical results

3.1 Threshold analysis results

Based on the above derived density evolution algorithm, the maximum correctable burst erasure lengths for the proposed 2D-SC-LDPC codes, SC-LDPC codes and ASC-LDPC codes can be computed. Set the maximum iteration number as 10^4 and the breakout condition of the erasure probability for each variable node as 10^{-6} . The stopping criterion is that the erasure probability for each variable node reaches the breakout condition or the iteration reaches the maximum iteration number.

As demonstrated in Ref. [10], the theoretical upper bound on the maximum correctable burst erasure length for the SC-LDPC code is the number of rows of its base matrix. Therefore, the Shannon limit e_{SL} can be defined as the number of rows of the base matrix. The $(3, 6, 3, 6, 30)$ 2D-SC-LDPC code is first considered and the base matrix has the size of 68×120 . The threshold analysis results obtained by the density evolu-

tion algorithm are shown in Table 1, which also lists the thresholds of the (3, 6, 30) ASC-LDPC code and the (3, 6, 30) SC-LDPC code for comparisons. The gap in Table 1 represents the difference between the Shannon limit e_{SL} and the threshold e_{BP} .

Table 1 Threshold results for different code ensembles

Code ensembles	e_{SL}	e_{BP}	gap
(3, 6, 3, 6, 30)2D-SC-LDPC code	68	59	9
(3, 6, 30)ASC-LDPC code	34	7	27
(3, 6, 30)SC-LDPC code	32	3	29

From Table 1, it can be observed that the gap to the Shannon limit of the 2D-SC-LDPC code is much smaller than the ASC-LDPC code and SC-LDPC code with the same coupling length and degree distributions. The ASC-LDPC code benefiting from its asymmetric structure is slightly better than the conventional SC-LDPC code. The results show that the proposed 2D-SC-LDPC codes are more robust against burst erasures than ASC-LDPC codes and SC-LDPC codes.

To investigate it more thoroughly, the thresholds for different 2D-SC-LDPC code ensembles with different degree distributions are also compared. The (4, 8, 4, 8, 30) 2D-SC-LDPC code and the (5, 10, 5, 10, 30) 2D-SC-LDPC are taken into consideration. Their threshold analysis results are shown in Table 2 and Table 3 respectively.

Table 2 Threshold results for different code ensembles

Code ensembles	e_{SL}	e_{BP}	gap
(4, 8, 4, 8, 30)2D-SC-LDPC code	72	59	13
(4, 8, 30)ASC-LDPC code	36	9	27
(4, 8, 30)SC-LDPC code	33	3	30

Table 3 Threshold results for different code ensembles

Code ensembles	e_{SL}	e_{BP}	gap
(5, 10, 5, 10, 30)2D-SC-LDPC code	76	59	17
(5, 10, 30)ASC-LDPC code	38	11	27
(5, 10, 30)SC-LDPC code	34	3	31

From Table 2 and Table 3, it can be seen that the thresholds of the proposed 2D-SC-LDPC codes are much better than the ASC-LDPC codes and SC-LDPC codes. Moreover, the maximum correctable burst erasure lengths are almost the same for different 2D-SC-LDPC codes with different degree distributions, which indicates that the 2D-SC-LDPC codes with low degrees are good enough for combating burst erasures.

The 2D-SC-LDPC code proposed in this paper has stronger robustness in the face of burst erasures than

ASC-LDPC codes and SC-LDPC codes, and its superior performance is due to two reasons. First, the ASC-LDPC coupling chain is used in each dimension during the construction of the 2D-SC-LDPC code, because the base matrices of the ASC-LDPC codes possess larger minimal cardinality of stopping sets, which is beneficial for improving the error correction capability of burst erasures and making it more robust to burst erasures than the conventional SC-LDPC code. The 2D-SC-LDPC code constructed with the asymmetric structure is more robust in the face of burst erasures. Second, it benefits from the two-dimensional construction method, which is a parallel connection of two ASC-LDPC coupling chains by means of edge exchanges. When faced with a burst erasure, the two coupling chains can provide information support to each other and avoid the situation that the decoding terminates when one of the coupling chains undergoes a burst erasure and thus enhances the ability to combat burst erasures. At the same time, it does not change the degree distribution of each coupling chain itself and also does not increase the decoding complexity.

3.2 Finite length performances

To confirm the asymptotic performances obtained by density evolution, the finite-length performances of the 2D-SC-LDPC codes over the burst erasure channels are simulated. First the proposed (3, 6, 3, 6, 30, 50) 2D-SC-LDPC code is compared with the (3, 6, 30, 50) ASC-LDPC code and (3, 6, 30, 50) SC-LDPC code. The code lengths of these three codes are 6000, 3000, 3000 and the design rates are 0.4333, 0.4333, 0.4667. The code lengths are computed by the denominators of Eq. (7), Eq. (5) and Eq. (3) respectively. The design rates are obtained by Eq. (7), Eq. (5) and Eq. (3) respectively. To further investigate the performance for resistance to the burst erasures, the SC-GLDPC codes is also considered for comparison. SC-GLDPC codes combat the burst erasures by introducing the linear block codes with error-correcting capabilities to replace the traditional single parity codes without error-correcting capabilities. Specifically, consider a (2, 7, L, M) SC-GLDPC code with choosing the (7, 4) Hamming code as the component code. Assume the coupling length $L = 30$ and the lifting factor $M = 30$ and then the code length is 6300. The finite-length performance of the 2D-SC-LDPC code transmitted over the burst erasure channels is shown in Fig. 4, along with the simulation results of the SC-GLDPC code, ASC-LDPC code and the SC-LDPC code.

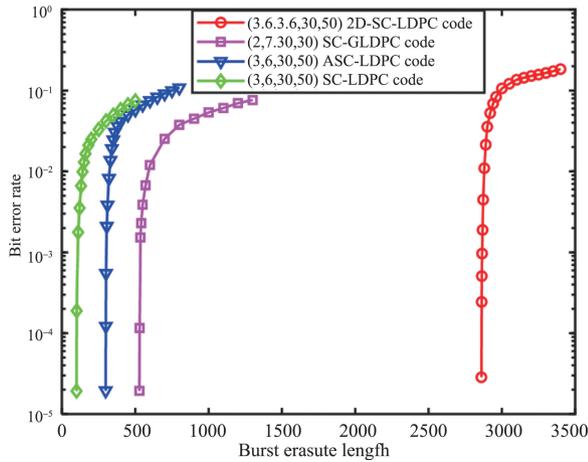


Fig. 4 Finite-length comparisons between the (3, 6, 3, 6, 30, 50) 2D-SC-LDPC code, the (2, 7, 30, 30) SC-GLDPC code, the (3, 6, 30, 50) ASC-LDPC code and the (3, 6, 30, 50) SC-LDPC code over the burst erasure channels

From Fig. 4, it can be observed that the starting points corresponding to the waterfall region of the 2D-SC-LDPC code, ASC-LDPC code and SC-LDPC code are about 2944, 346 and 145 respectively. Since the iterative decoding threshold is well known that it corresponds to the starting point of the waterfall region in the BER curve, the starting points of the waterfall region can be estimated by the threshold e_{BP} multiplying the lifting factor M , that is $e_{BP} \times M$. Therefore, according to the threshold results shown in Table 1, the estimated starting points can be obtained and they are 2950, 350 and 150 respectively. The comparison results indicate that the finite-length simulation results are roughly consistent with the threshold analysis results. As for the simulation results, it can be seen that although the codelength of the 2D-SC-LDPC code is twice as long as the ASC-LDPC code and the SC-LDPC code, the maximum correctable burst erasure length of the proposed 2D-SC-LDPC code is approximately eight times larger than ASC-LDPC code and twenty times larger than SC-LDPC code, which indicates that the capability of correcting burst erasures can be improved dramatically by the proposed code structure. One possible reason is that the edge exchange is introduced in constructing the 2D-SC-LDPC code, which can provide the information support to each other when either of them faces with the burst erasures. Besides, each dimension adopts the asymptotic structure in the construction of 2D-SC-LDPC code. Since the base matrices of the ASC-LDPC code possess larger minimal cardinality of stopping sets, it is beneficial for improving the error correction capability of burst erasures. Moreover, the perform-

ances of the proposed 2D-SC-LDPC codes for combating the burst erasures are much better than the SC-GLDPC codes in the case of a comparable code lengths as shown in Fig. 4.

In the following, the finite-length performances of the (4, 8, 4, 8, 30, 100) 2D-SC-LDPC code and the (5, 10, 5, 10, 30, 100) 2D-SC-LDPC code are also simulated. The comparison results of the proposed 2D-SC-LDPC codes with the ASC-LDPC codes and SC-LDPC codes are shown in Fig. 5 and Fig. 6.

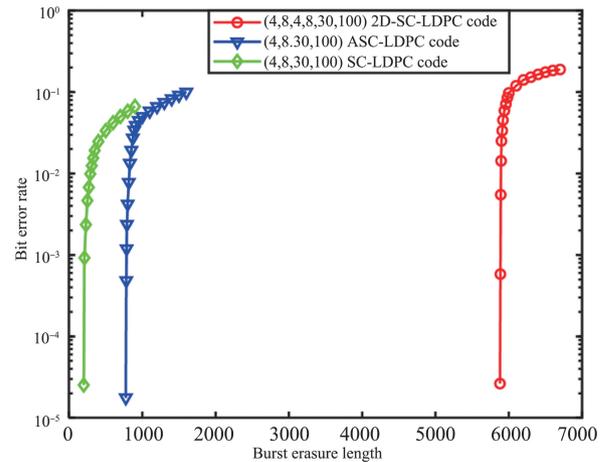


Fig. 5 Finite-length comparisons between the (4, 8, 4, 8, 30, 100) 2D-SC-LDPC code, the (4, 8, 30, 100) ASC-LDPC code and the (4, 8, 30, 100) SC-LDPC code over the burst erasure channels

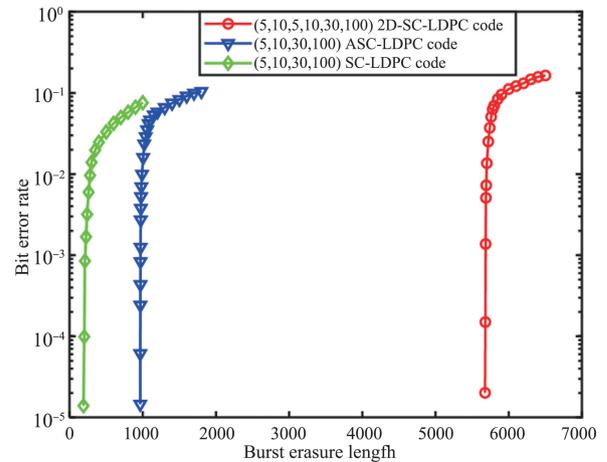


Fig. 6 Finite-length comparisons between the (5, 10, 5, 10, 30, 100) 2D-SC-LDPC code, the (5, 10, 30, 100) ASC-LDPC code and the (5, 10, 30, 100) SC-LDPC code over the burst erasure channels

From Fig. 5, it can be observed that the starting points corresponding to the waterfall region of these three codes are 5898, 897, 295 and the predicted ones are 5900, 900, 300, which shows that the finite-length performances for resistance to burst erasures can fit well with the asymptotic performances obtained by den-

sity evolution. As for the (5, 10, 5, 10, 30, 100) 2D-SC-LDPC code, similar results can be concluded, where the starting points corresponding to the waterfall region of these three codes are 5890, 1094, and 296 in simulations while the estimated ones are 5900, 1100 and 300. From the above simulation results, it can be observed that the proposed 2D-SC-LDPC codes perform better than the ASC-LDPC codes and the SC-LDPC codes for combating burst erasures either in the asymptotic performances or the finite-length performances.

4 Conclusion

In this paper, a 2D-SC-LDPC codes to combat the burst erasures is proposed. To evaluate the performances against burst erasures, the density evolution algorithm for 2D-SC-LDPC codes over the burst erasure channels is proposed to compute the maximum correctable burst erasure length. Density evolution analysis shows when transmitted over the burst erasure channel, the proposed 2D-SC-LDPC codes are more robust against burst erasures than the ASC-LDPC codes and SC-LDPC codes. Moreover, the finite-length performances of the proposed 2D-SC-LDPC codes along with the ASC-LDPC codes and SC-LDPC codes over the burst erasure channels are also simulated to confirm the asymptotic performances obtained by density evolution. Simulation results show that the finite-length performances of the proposed 2D-SC-LDPC codes are consistent with the asymptotic performances and they perform much better than ASC-LDPC codes and SC-LDPC codes.

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