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Stability analysis via the concept of Lyapunov exponents — a case study①

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Abstract

The dynamics characteristics of the robotic arm system are usually highly nonlinear and strongly coupling, which will make it difficult to analyze the stability by the methods of solving kinetic equations or constructing Lyapunov function, especially, these methods cannot calculate the quantitative relationship between mechanical structures or control input and dynamics parameters and stability. The theoretical analysis process from symbol dynamics modeling of the robotic arm system to the movement stability is studied by using the concept of Lyapunov exponents method. To verify the algorithm effectiveness, the inner relation between its joint input torque and stability or chaotic and stable motion of the 2-DOF robotic arm system is analyzed quantitatively. As compared with its counterpart of Lyapunov's direct method, the main advantage of the concept of Lyapunov exponents is that the methods for calculating the exponents are constructive to provide an effective analysis tool for analyzing robotic arm system movement stability of nonlinear systems.

Key words: stability analysis, nonlinear systems, Lyapunov exponents (LEs), 2-DOF robotic arm system

0 Introduction

Lyapunov's stability theory is of central importance for stability analysis of nonlinear systems, especially of robot control systems. The classical approach to handle the stability and stabilization issues for dynamical systems is based on constructing a Lyapunov function which satisfies sufficient conditions and guarantees stability. However, there is no constructive method available for such derivation. Consequently, stability of many nonlinear systems cannot be analyzed.

An alternative tool for the stability analysis of the dynamical systems is the concept of Lyapunov exponents (LEs). LEs, defined as the average exponential rates of divergence or convergence of nearby orbits in the state space, can indicate system stability[1-3]. It was first introduced by Lyapunov to study the stability of non-stationary solution of ordinary differential equations[4]. It is a powerful tool to categorize the steady-state behavior of dynamic systems. A LE is a number that reflects the averaged exponential rate of divergence or convergence of nearby orbits in the state space. Generally the sum of all the LEs represents the average volume contraction / expansion rate in a state space, and the signs of LEs indicate the asymptotic property of the dynamical system (Williams 1997)[3]. Any attractor of a dissipative system will have at least one negative exponent and the sum of all exponents is negative. More detailedly speaking, in a dissipative system, an attractor is defined to be chaotic if the spectrum of LEs contains at least one positive exponent. For non-chaotic attractors such as periodic or quasi-periodic ones, there are only zero and negative exponents, while those exponentially stable equilibrium points are those characterized by all LEs being negative. The method for calculating the LEs is constructive for any dynamic systems. This constructive nature makes it more advantageous over Lyapunov's direct method.

For complicated systems, it is in generally impossible to determine LEs analytically. LEs are usually calculated numerically using a mathematical model. The results can characterize the system stability provided that the numerical artifact is under control[5]. Methods for calculating LEs based on a mathematical model have been well developed[6,7] and widely used for diagnosing chaotic systems as well as stability analysis of

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complex nonlinear systems. For example, using LEs, Asokanthan and Wang[8] studied the torsional stability of a Hooke's joint driven system, and Gilat and Aboudi[9] studied parametric stability of nonlinearly elastic composite plates. Zevin and Pinski[10] developed absolute stability criterion for a non-autonomous linear system controlled by a nonlinear feedback control with a time-varying delay based on LEs. Awrejcewicz and Kudra[11] carried out the stability analysis of a multibody mechanical system with rigid unilateral constraints via LEs. Rogelio, et al[12]. developed a method for nonlinear analysis of boiling water sector in a nuclear generator based on LEs. Wu, et al[13-18]. developed a series of stability analysis of biped standing via the concept of LEs. Liu[19,20] researched the stability analysis of unmanned aerial vehicle using the LEs. Therefore, the theoretical analysis process from the symbol dynamics modeling of the robot system to the movement stability is studied using the concept of LEs method. To verify the algorithm effectiveness, the inner relation between its joint input torque and stability of the 2-DOF robotic arm system is analyzed quantitatively, which provides an effective analysis tool to analyze the robot movement stability of nonlinear system.

1 Mathematical preliminary

Here, first the concept of Lyapunov exponents (LEs) is reviewed, followed by a brief description of procedures of calculating LEs based upon mathematical models.

1.1 Concept of Lyapunov exponents

LEs $\lambda_i (i = 1, \cdots, n)$ are the average exponential rates of divergence or convergence of nearby orbits in the state space. Wolf, et al[1]. (1985) defined a spectrum of LEs in the manner most relevant to spectral calculations. Considering a continuous dynamic system in an n-dimensional state space, this concept monitors the long-term evolution of an infinitesimal n-sphere of initial conditions. Due to the dynamic flow, the n-sphere may be deformed to an n-ellipsoid as graphically shown in Fig. 1 when $n = 2$. The average rates of the length expanding or contracting of the ellipsoid principal axes over an infinite time period are called LEs. The ith-dimensional LE is then defined in terms of the length of the ellipsoidal principal axis $\| \delta x_i(t) \|$:

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\| \delta x_i(t) \|}{\| \delta x_i(t_0) \|}, \quad (i = 1, \cdots, n) \quad (1)$$

where $\| \delta x_i(t_0) \|$ and $\| \delta x_i(t) \|$ represent the lengths of the ith principal axis of the infinitesimal n-dimensional hyper-ellipsoid at initial and current time instances, t_0 and t, respectively. Since one LE can be defined for each principal axis, the total number of system LEs is equal to the dimension of the dynamic system. Although in the calculation of LEs choosing a trajectory (the 'fiducial' trajectory) is needed, the consequence of a theorem of Oseledec[7] proves that LEs are global properties of the dynamic systems and independent of the chosen trajectory ('invariant' measure of the dynamic system). It is important to note that the orientation of the ellipsoid changes continuously as it evolves. Therefore, it is not possible to define the direction associated with a given exponent.

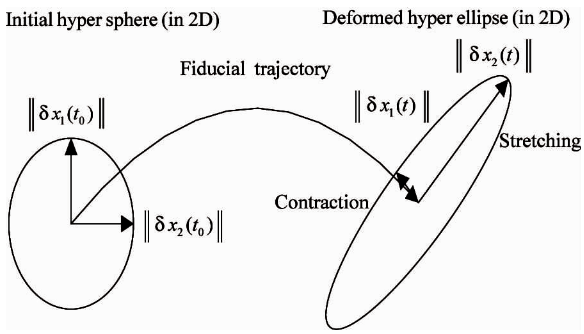


Fig. 1 Evolution of infinitesimal two-dimensional sphere of initial conditions

Both the signs and values of a system LE have information about exponential behavior of the dynamic systems. The signs of LEs reveal the stability property of the system's dynamics. Negative exponents correspond to those principal axes of the ellipsoid that shrink in average. If all the exponents are negative, the dynamic system is exponentially stable and the attractor is a fixed point (equilibrium point). Zero exponents indicate slow change in magnitudes of the principal axes. A system with one zero exponent and other negative ones has a one-dimensional attractor. For systems with order 3 or more, the positive LEs indicate chaotic behavior. In a chaotic system, the long-term behavior of an initial condition that is specified with any uncertainty cannot be predicted[1]. The sum of all LEs indicates the time averaging divergence of the phase space volume; hence, for any dissipative dynamic systems, the sum of all exponents is negative[1], which implies that dissipative systems have at least one negative exponent. Moreover, dissipative systems with no fixed point must have at least one zero exponent[21].

In general, there is no feasible analytical way to determine the LEs for a complicated system[22]. Therefore, the LEs are often computed numerically. They can be computed using either the system mathematical model or a time series. In practical applications, the finite-time LEs are frequently used in the following form:

$$\lambda_i \approx \frac{1}{t} \ln \frac{\| \delta x_i(t) \|}{\| \delta x_i(t_0) \|}, \quad (i = 1, \cdots, n) \qquad (2)$$

in the limit as $t \to \infty$, the finite-time LEs converge to the true LEs[23].

1.2 Calculation of Lyapunov exponents using standard algorithm

The standard algorithm for calculating the spectrum of LEs from explicit mathematical models of systems was developed by Wolf, et al[1]. (1985). In Wolf's pioneer work a fiducial trajectory (the centre of the sphere) is defined by the action of the nonlinear motion equations on some initial conditions. The principal axes are determined by the evolution via the linearised equations of an initially orthonormal vector frame anchored to the fiducial trajectory. This leads to the following set of equations (Wolf, et al. 1985):

$$\begin{Bmatrix} \dot{x} \\ \dot{\psi}_t \end{Bmatrix} = \begin{Bmatrix} f(x) \\ F(t)\psi_t \end{Bmatrix} \qquad (3)$$

where ψ_t is called the state transition matrix of the linearized system $\delta x(t) = \psi_t \delta x_0$ and the Jacobian $F(t)$ is defined as

$$F(t) = \frac{\partial f(x)}{\partial x^{\mathrm{T}}} \Big|_{x = x(t)} \qquad (4)$$

And the initial conditions for numerical integrations are

$$\begin{Bmatrix} x(t_0) \\ \psi_t(t_0) \end{Bmatrix} = \begin{Bmatrix} x_0 \\ I \end{Bmatrix} \qquad (5)$$

where I is the identity matrix of proper dimension.

To avoid misalignment of all the vectors δx_i along the direction of maximal expansion, they are reorthonormalised at each integration step by involving the Gram-Schmidt reorthonormalisation (GSR) scheme, which generates an orthonormal set $\{u_1, \cdots, u_n\}$ of n vectors with the property that $\{u_1, \cdots, u_n\}$ spans the same subspace as $\delta x_1, \cdots, \delta x_n$. This orientation-preserving property of GSR suggests that the initial labeling of the vectors may be done arbitrarily. Fig. 2 shows the geometrical interpretation of the orthonormalisation for two principal axes at the j-th step. Once the orthonormal vector frame $\{u_1, \cdots, u_n\}$ is produced by GSR, for large enough integer K, one can obtain Lyapunov exponents as follows with time-step size h properly chosen:

$$\lambda_i \approx \frac{1}{Kh} \sum_{j=1}^{k} \ln \| u_i^{(j)} \|, \quad (i = 1, 2, \cdots, n) \qquad (6)$$

where j is the number of integration steps.

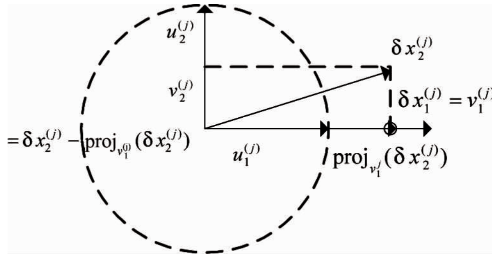


Fig. 2 The geometrical interpretation of GSR for $\delta x_1^{(j)}$ and $\delta x_2^{(j)}$ ($j = 1, \cdots, K$ and j is the number of integration step). $\delta x_1^{(j)}$ and $\delta x_2^{(j)}$ are orthogonalised into $v_1^{(j)}$ and $v_2^{(j)}$, and then normalised into $u_1^{(j)}$ and $u_2^{(j)}$. Here $proj_{v1(j)}(\delta x_2^{(j)})$ denotes the projector of the vector $\delta x_2^{(j)}$ on the vector $v_1^{(j)}$. And $v_2^{(j)}$ is exactly equal to $\delta x_2^{(j)} - proj_{v1(j)}(\delta x_2^{(j)})$.

Overall, the calculation of the Lyapunov exponents from the system's mathematical model can be shown in the following step:

Step 1 Establish dynamics model, with the standard form as:

$$\dot{q} = V(q)p$$
$$M(q)\dot{p} + C(q,p)p + F(p,q,u) = 0$$

Step 2 Transform model equation into state equation

$$\frac{\mathrm{d}X(t)}{\mathrm{d}t} = f(X(t))$$

Step 3 Calculate Jacobian

$$| \mathrm{d}f(X)/\mathrm{d}X |_{X_i}$$

Step 4 Calculate Lyapunov exponents

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln | \mathrm{d}f(X)/\mathrm{d}X |_{X_i}$$

2 2-DOF robotic arm

In this study, the 2-DOF robotic arm system, the basic unit in the chain robot system, playing an important role in the field of industrial production is shown in Fig. 3. A 2-DOF robotic arm system is a pendulum with

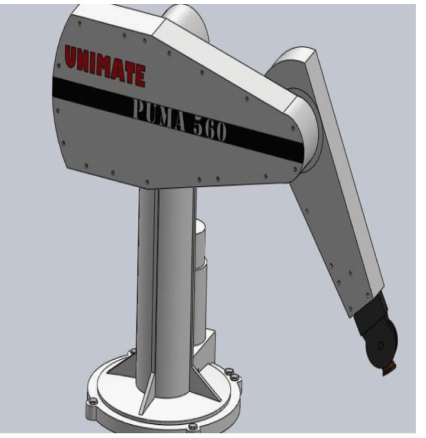


Fig. 3 PUMA560 robot

a second pendulum attached to its end, exhibiting rich dynamic behaviors. As shown in Fig. 4. The pendulum system consists of two rigid links with length l_1 and l_2. The base of the system is fixed. m_1 and m_2, are the masses of the two links. r_1 and r_2, are the locations of the mass centers of the two links. θ_1 and θ_2, are the joint angles. T_1 and T_2, are the control torques applied at both joints.

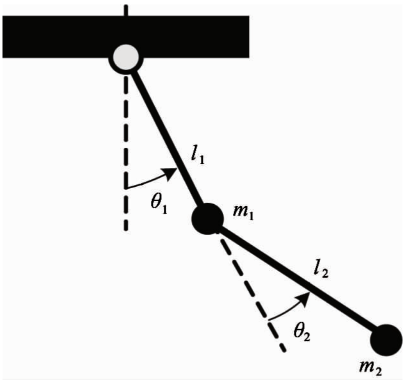


Fig. 4 2-DOF robotic arm model

The 2-DOF robotic arm model has 4 states. In the this case the model can be generated in the form of Poincare's equation[24]:

$$\dot{q} = V(q)p \tag{7}$$

$$M(q)\dot{p} + C(q, p)p + F(p, q, u) = 0 \tag{8}$$

where $q = (\theta_1, \theta_2)^T$ is the generalized coordinate vector, $p = (w1x, w2x)^T$ is the quasi-velocity vector. Key parameters in the formulation are: kinematic matrix $V(q)$, inertial matrix $M(q)$, gyroscopic matrix $C(q, p)$, and force function $F(p, q, u)$ includes all of the gravitational forces and moments. Kinematic matrix in Eq. (7) is of the

$$V(q) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The parameters in dynamics of Eq. (8) are:

$$M(q) = \begin{bmatrix} J_1 + J_2 + 2\kappa\cos(\theta_2) & J_2 + \kappa\cos(\theta_2) \\ J_2 + \kappa\cos(\theta_2) & J_2 \end{bmatrix}$$

$$F(p, q, u) = \begin{bmatrix} -T_1 - g(-(l_1 m_2 + m_1 r_1)\sin\theta_1 - m_2 r_2 \sin(\theta_1 + \theta_2)) \\ -T_2 + g m_2 r_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$C(p, q) = \begin{bmatrix} 0 & -\kappa w1x Sin(\theta_2) - 2\kappa w1x Sin(\theta_2) \\ \frac{1}{2}(\kappa w2x\sin(\theta_2) + 2\kappa w1x\sin(\theta_2)) & -\frac{1}{2}\kappa w1x Sin(\theta_2) \end{bmatrix}$$

where, $J_1 = I_1 + m_1 \cdot r_1^2$, $J_2 = I_2 + m_2 \cdot r_2^2$, $\kappa = m_2 \cdot l_1 \cdot r_2$.

And the proportional and derivative (PD) controller will be used to control the two-link pendulum system having stable and chaotic motion, respectively. For the PD controller, the torque at each joint is

$$T_i = k_{pi}(\theta_{di} - \theta_i) + k_{vi}(\dot{\theta}_{di} - \dot{\theta}_i), \ i = 1, 2 \tag{9}$$

where θ_{di} is the desired position or a periodic trajectory to be tracked at each joint, k_{pi} and k_{vi} are the positive proportional and derivative gains. By changing the control gains k_{pi} and k_{vi}, the system can exhibit either chaotic motion or stable motion. For the stable motion, two simulations are carried out. One is to keep the pendulum system at a set point, and the other is to track a desired motion. For the chaotic motion, the system is intended to track the desired trajectory. The parameters of the two link pendulums are as follows:

Table 1 Parameters of the two-link pendulums

Link	Length (m)	Mass (kg)	Mass centers of link (m)	Inertia ($kg \cdot m^2$)
1	0.5	20	0.2	6
2	0.4	8	0.3	1.5

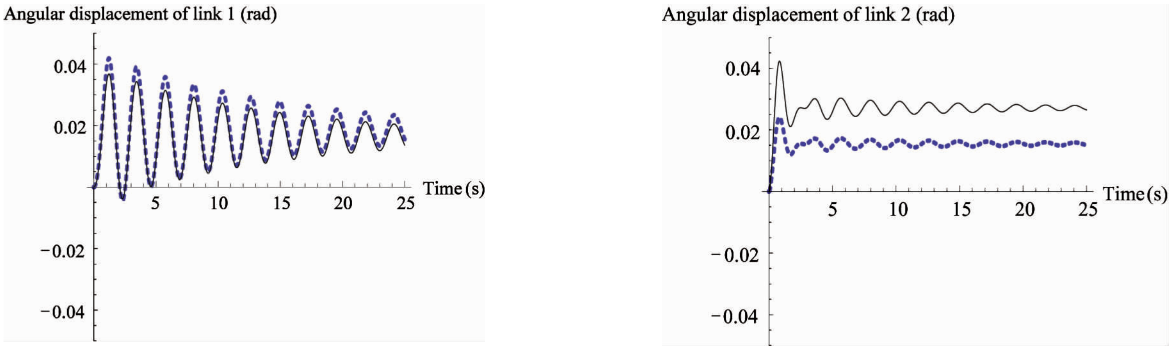
3 Stability analysis of the 2-DOF robotic arm

3.1 Simulation of chaotic motion

The desired trajectories are $\theta_{d1} = 0.57 \times \sin(2\theta_1)$, $\theta_{d2} = 0.5 \times 0.57 \times \sin(2\theta_2)$, controller gains are selected as $k_{pi} = 10(N/\text{rad})$, $k_{vi} = 2(N \times S/\text{rad})$, where $i = 1, 2$, and initial conduction $\{\theta_1 = 0(\text{rad}),\ w_{1x} = 0(\text{rad / s}),\ \theta_2 = 0(\text{rad}),\ w_{2x} = 0(\text{rad/s})\}$. It can be seen easily that link 1 and link 2 do not follow the desired trajectories in Fig. 5. The solid line and dash line are respectively actual angular displacement and desired angular displacement. The initial angular displacement is changed $\theta_1 = \theta_2 = 0.01(\text{rad})$. The result of the attractor is changed significantly, which demonstrates the system is sensitive to the initial condition in Fig. 6. In addition, the LEs for chaotic motion from the mathematic mode are shown in Fig. 7. All exponents converging to constants, the largest LEs greater 0 and the motion is indeed chaotic.

3.2 Simulation of stable tracking motion

By changing the gains in the PD controllers, the 2-DOF robotic arm system can exhibit the stable motion. In the case of stable tracking motion that the system follows, the desired trajectory is also the $\theta_{d1} = 0.57 \times \sin(2\theta_1)$, $\theta_{d2} = 0.5 \times 0.57\sin(2\theta_2)$, the controller was chosen as



(a) link 1

(b) link 2

Fig. 5 The chaotic motion for the link 1 and 2

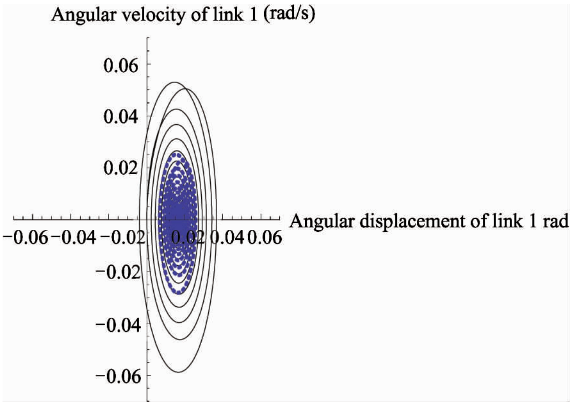


Fig. 6 Attractor of chaotic motion link 1

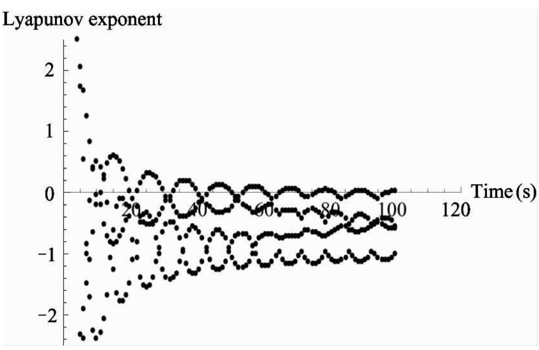
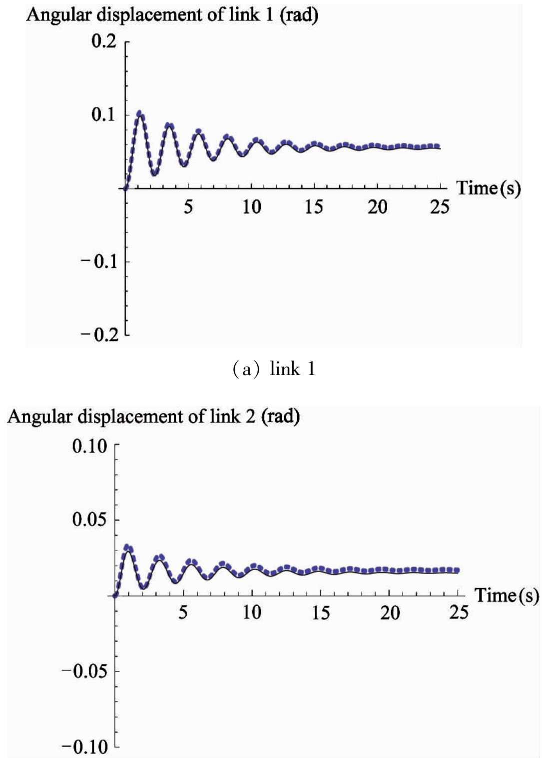


Fig. 7 The Lyapunov Exponents for chaotic motion



(a) link 1

(b) link 2

Fig. 8 The stable tracking motion for the link 1 and 2

$k_{p1}=30(N/\mathrm{rad})$, $k_{v1}=5(N\times S/\mathrm{rad})$, $k_{p2}=12(N/\mathrm{rad})$, $k_{v2}=3(N\times S/\mathrm{rad})$ in the simulations. The initial conduction is also $\{\theta_1=0(\mathrm{rad}), w_{1x}=0(\mathrm{rad/s}), \theta_2=0(\mathrm{rad}), w_{2x}=0(\mathrm{rad/s})\}$. The dash lines are the desired trajectories and the solid lines are the actual trajectories. Fig. 8 shows link 1 and link 2 stable tracking motion. That is, the 2-DOF robotic arm system can successfully follow the desired trajectory by using suitable controller gains. Correspondingly, by changing the initial angular displacement $\theta_1=\theta_2=0.01(\mathrm{rad})$, the attractor is shown in Fig. 9. Similar to the chaotic motion, there are four negative exponents for the stable tracking, and four negative exponents indicate that the trajectories converge to the desired (See Fig. 10).

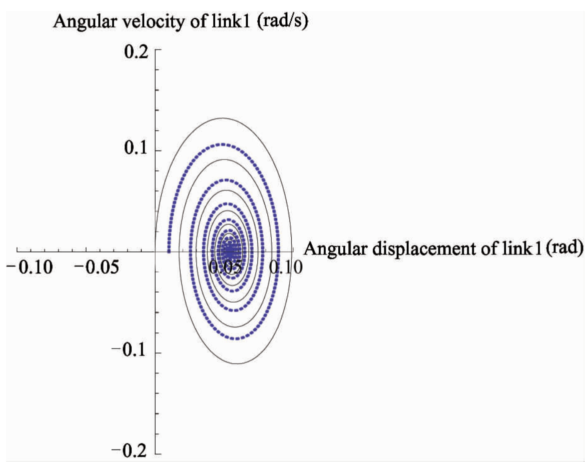


Fig. 9 Attractor of stable tracking motion

Meanwhile, the second case for the simulation of stable motions is that the robotic arm system is controlled to approach the set point. The gains in the PD controller are still set as $k_{p1} = 30(N/\text{rad})$, $k_{v1} = 5(N \times S/\text{rad})$, $k_{p2} = 12(N/\text{rad})$, $k_{v2} = 3(N \times S/\text{rad})$. The system will start from the initial condition $\{\theta_1 = 1(\text{rad}),\ w_{1x} = 0(\text{rad/s}),\ \theta_2 = 1(\text{rad}),\ w_{2x} = 0(\text{rad/s})\}$, then controlled by the PD controller to approach point $\{\theta_1 = 0(\text{rad}),\ w_{1x} = 0(\text{rad/s}),\ \theta_2 = 0(\text{rad}),\ w_{2x} = 0(\text{rad / s})\}$. The angular displacements of link 1 and link 2 are simulated shown in Fig. 11. The equilibrium point approximately $\{0, 0, 0, 0\}$ is shown in the phase space in Fig. 12. The LEs for stable motion with a set point are shown in Fig. 13.

Summarily, the angular displacement of each link is simulated. The attractors of chaotic, stable motion are also shown. Results show that the robotic arm system can exhibit different motions under different control parameters. Also it is valid by using the concept of LEs.

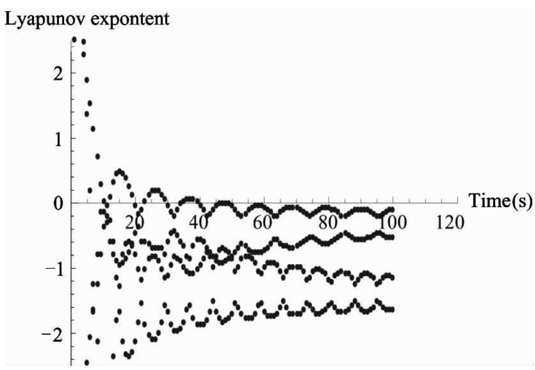


Fig. 10 Lyapunov exponents for stable tracking motion

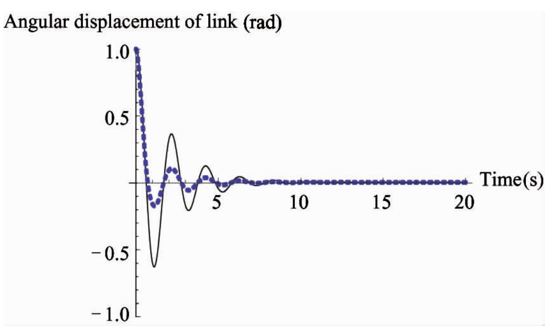
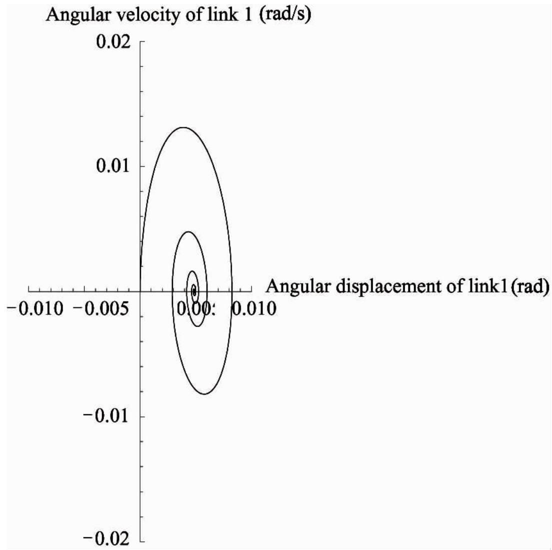


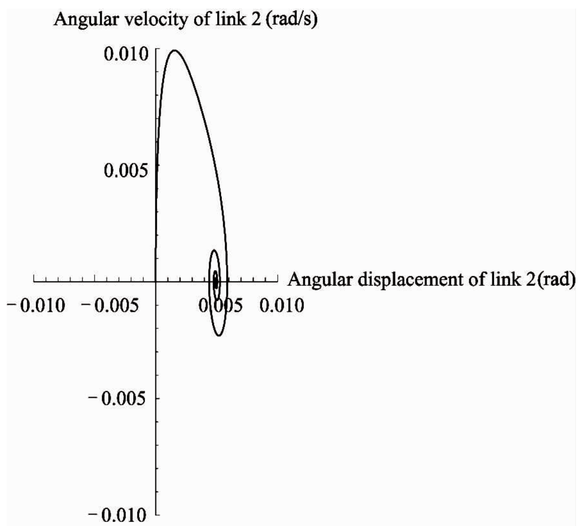
Fig. 11 Stable response with a set point

4 Conclusions

Two different types of stability have been rigorously analyzed in this article using the concept of LEs for a 2-DOF robotic arm model: chaotic motion with disturbance imposed on the initial conditions, and stable motion with changing control gains. Moreover, the LEs



(a) link 1



(b) link 2

Fig. 12 Attractor response for the stable motion with a set point

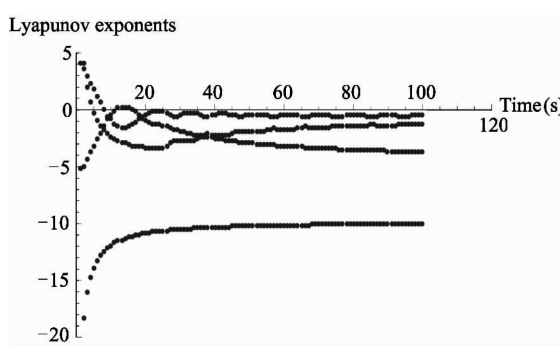


Fig. 13 Lyapunov Exponents for stable motion with a set point

are an 'invariant' measure of the dynamic systems, that is, the LEs are independent of initial conditions. The main contribution of this article lies in the fact that, starting from the symbol dynamics modeling to stability analysis the concept of LEs is used to change the controller gain and obtain different states. As discussed earlier, the presented method for calculating LEs is applicable to general dynamic systems provided that the mathematical models are available. Therefore, the proposed method can be extended to more complex models, especially to multi-body dynamics, which is the subject of future work.

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